Journal of Global Optimization **8:** 379–391, 1996.

An Algorithm for Maximizing a Convex Function over a Simple Set

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(Received: 19 October 1993; accepted: 16 October 1995)

Abstract. The problem of maximizing a convex function on a so-called simple set is considered. Based on the optimality conditions [19], an algorithm for solving the problem is proposed. This numerical algorithm is shown to be convergent. The proposed algorithm has been implemented and tested on a variety of test problems.

Key words: Global optimization, optimality condition, convex function, numerical algorithm, simple set.

1. Introduction

In this paper we consider the problem of maximizing a convex function on a subset $D \subset \mathrm{R}^n$:

$$f(x) \to \max, \qquad x \in D. \tag{1}$$

The first work devoted to the solution of the problem of maximizing a convex function on a convex set appeared more than twenty years ago [21]. This problem belongs to a class of global optimization problems that have many practical applications, including allocation problems and problems on the optimal regime of power systems [10,16]. In addition, the problem of the global minimization of the difference $g - h$ of two convex functions g and h can be transformed into problem (1) with a suitable convex set D [5].

There are many theoretical and numerical papers [2,5,6,7,8,13,14,18,19,22,23] devoted to the solution of problem (1). When $f(x)$ is convex, quadratic algorithms are described in [9,20]. This paper is organized as follows. In Section 2, we derive the optimality condition for problem (1) and we introduce a definition of a "simple" set. In Section 3, we present an algorithm to solve the problem of maximizing a convex function on a "simple" set and we show its convergence. In Section 4, we present some computational results obtained with our algorithm on test problems.

2. Formulation of the Problem and Constructive Form of Optimality Conditions

We consider the problem

$$f(x) \to \max, \qquad x \in D \subset \mathrm{R}^n, \tag{2}$$

where $f : \mathrm{R}^n \to \mathrm{R}$ is a convex and twice differentiable function and D is an arbitrary subset of R^n.

It is well known [20] that, if the function f is nonconstant, then the optimality conditions for problem (2) will be given by the following theorem:

THEOREM 1. *A point x^* is a solution of problem (2) if and only if the following condition holds*

$$\begin{cases} \forall y: \quad f(y) = f(x^*) \\ \langle f'(y), x - y\rangle \le 0, \quad \forall x \in D. \end{cases} \tag{3}$$

(Here and in the following $\langle , \rangle$ denotes the scalar product of two vectors.)

Proof. Necessity. Assume that x^* is a solution of problem (2). Let the points x and y be such that

$$\forall y \in \mathrm{R}^n : f(y) = f(x^*), \qquad x \in D.$$

Then, by the convexity of f, we have

$$0 \ge f(x) - f(x^*) = f(x) - f(y) \ge f'(y), x - y \ge .$$

Sufficiency. Suppose that x^* is not a solution for problem (2), i.e.,

$$\exists u \in D : f(u) > f(x^*).$$

Now we introduce the closed and convex set:

$$C = \{x \in \mathrm{R}^n / f(x) \le f(x^*)\}.$$

Note that $u \notin C$. Then there exists the projection of the point u on C, i.e.,

$$\exists y \in C : \|y - u\| = \inf_{x \in C} \|x - u\|.$$

By construction we have:

$$\|y - u\| > 0. \tag{4}$$

On the other hand y can be considered as a solution of the following convex programming problem:

$$g(x) = \frac{1}{2}\|x - u\|^2 \to \min, \qquad x \in C.$$

Let us write down the optimality conditions for this problem at the point y as follows:

$$\begin{cases} \exists\lambda_0 \geq 0, \quad \lambda \geq 0, \quad \lambda_0 + \lambda > 0 \\ \lambda_0 g'(y) + \lambda f'(y) = 0 \\ \lambda(f(y) - f(x^*)) = 0. \end{cases} \tag{5}$$

If $\lambda_0 = 0$ then (5) implies that $\lambda > 0$, $f'(y) = 0$ and $f(y) = f(x^*)$.

This is impossible because it contradicts the statement that the function f is nonconstant. The case $\lambda = 0$ is also impossible, because $g'(y) = 0$ and it contradicts (4).

Taking into account that $g'(y) = y - u$ from (5) we have:

$$\begin{aligned} & y - u + \lambda f'(y) = 0, \quad \lambda > 0, \\ & f(y) = f(x^*) \\ & \langle f'(y), u - y \rangle > 0, \end{aligned}$$

which contradicts condition (3).

This contradiction implies that the assumption that x^* is not a solution of problem (2) must be false. This completes the proof. □

We introduce the following definition.

DEFINITION 1. A set D is a simple set if the following conditions hold:

(a) D is compact.

(b) The problem of maximizing a linear function on D is solvable using a 'simple method'.

We say that a method is simple if involves the use of a simplex method or the use of a method that gives an analytical solution to the problem of maximizing a linear function on D.

In addition, throughout this paper we assume that D is simple and also that the function f is strongly convex.

Let f^* denotes a global maximum of problem (2):

$$f^* \triangleq \max_{x \in D} f(x)$$

We define the auxiliary function $\Pi(y)$ in the following way:

$$\Pi(y) = \max_{x \in D} \langle f'(y), x - y \rangle \qquad \text{for all} \quad y \in R^n.$$

We present some of properties of the function $\Pi(y)$ without proof.

Detailed proofs are given in [20].

LEMMA 1. *$\Pi(y)$ is continuous on* R^n.

LEMMA 2. *There exists the directional derivative of* $\Pi(y)$ *at* y *in the direction* $h \in \mathrm{R}^n$ *which is given by the following formula:*

$$\frac{\partial \Pi(y)}{\partial h} = \langle f''(y)h, z\rangle - \langle f''(y)y + f'(y), h\rangle,$$

where z *is such that:*

$$\langle f'(y), z\rangle = \max_{x\in D}\langle f'(y), x\rangle.$$

Let us introduce the function $\theta(x)$ as follows:

$$\theta(x) = \max_{f(y)=f(x)} \Pi(y).$$

Note that, since $f(x)$ is strongly convex, then

$$\theta(x) < +\infty \qquad \text{for all} \quad x \in \mathrm{R}^n.$$

Using $\theta(x)$, we may show the following result.

THEOREM 2. *Let the function* f *be nonconstant on* R^n *and* $x^0 \in D$. *If* $\theta(x^0) \leq 0$ *then the point* x^0 *is a solution of problem (2).*

Proof. The proof is an obvious consequence of the inequality

$$\langle f'(y), x - y\rangle \leq \max_{x\in D}\langle f'(y), x - y\rangle \leq \theta(x^0) \leq 0,$$

which holds for all x and y such that: $x \in D, \quad f(y) = f(x^0)$.

In fact by Theorem 1 we have that x^0 is a solution of problem (2) and the proof is complete. □

Theorem 2 is used to verify the optimality condition (3).

3. Convergence of the Algorithm

In this section we present an algorithm to solve problem (2). In order to introduce our algorithm, we have to prove the following result.

THEOREM 3. *Let a sequence* $x^k \subset \mathrm{R}^n$ *be such that*

$$f(x^k) > f(x^{k-1}) > \ldots > f(x^0) \quad \textit{and} \quad x^0 \neq \arg\min_{x\in\mathrm{R}^n} f(x)$$

then $\exists\delta > 0 : \|f'(x^k)\| \geq \delta$ *for each* $k = 0, 1, 2 \ldots$.

Proof. Since f is strongly convex on R^n there exists a positive constant J such that, for all $x, y \in R^n$ and for all $\alpha \in [0, 1]$, the following inequality holds:

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y) - \alpha(1-\alpha)J\|x-y\|^2. \quad (6)$$

Now let x_* be the unique global minimizer of $f(x)$ on R^n:

$$f(x_*) = \min_{x \in R^n} f(x)$$

Then it is clear that $x^k \neq x_*$ for each $k = 0, 1, 2, \ldots,$.

From the convexity of f we have:

$$f(x) - f(y) \leq \langle f'(x), x - y\rangle \quad \text{for all} \quad x, y \in R^n. \quad (7)$$

If we substitute $x = x^k, \quad y = x_*$ and $\alpha = \frac{1}{2}$ into formulas (6) and (7) we obtain:

$$\tfrac{1}{4}J\|x^k - x_*\|^2 \leq \tfrac{1}{2}\left[f(x^k) - f(\tfrac{1}{2}x^k + \tfrac{1}{2}x_*)\right] +$$

$$+\tfrac{1}{2}\left[f(x_*) - f(\tfrac{1}{2}x^k + \tfrac{1}{2}x_*)\right] \leq \tfrac{1}{4}\langle f'(x^k), x^k - x_*\rangle +$$

$$+\tfrac{1}{4}\langle f'(x_*), x_* - x^k\rangle = \tfrac{1}{4}\langle f'(x^k) - f'(x_*), x^k - x_*\rangle.$$

Taking into account that $f'(x_*) = 0$ we have

$$J\|x^k - x_*\|^2 \leq \langle f'(x^k), x^k - x_*\rangle \leq \|f'(x^k)\|\|x^k - x_*\|.$$

Thus we get:

$$J\|x^k - x_*\| \leq \|f'(x^k)\| \qquad \text{for each} \quad k = 0, 1, 2 \ldots. \quad (8)$$

Moreover, by using the formulas (7) and (8) we find that:

$$0 < f(x^k) - f(x_*) \leq \langle f'(x^k), x^k - x_*\rangle \leq \|f'(x^k)\|\|x^k - x^*\| \leq \frac{1}{J}\|f'(x^k)\|^2$$

for all $k = 0, 1, 2 \ldots$.

Since the sequence $\{f(x^k)\}$ is strictly monotonically increasing, then the following inequality holds:

$$0 < J(f(x^0) - f(x_*)) \leq \|f'(x^k)\|^2$$

for each $k = 0, 1, 2 \ldots$.

Consequently, choosing $\delta = (J(f(x^0) - f(x_*)))^{\frac{1}{2}}$ the assertion is proved. □

ALGORITHM 1.

Step 1. Choose $x^0 \in D$ such that $x^0 \neq \arg\min_{x \in R^n} f(x)$. Set $k = 0$

Step 2. To determine the value of $\theta(x^k)$ solve the following constrained global optimization problem:

$$\Pi(y) \to \max, \quad f(y) = f(x^k).$$

Let y^k be a solution of this problem, i.e.,

$$\theta(x^k) = \Pi(y^k) = \max_{x \in D} \langle f'(y^k), x - y^k \rangle.$$

Moreover, the point x^{k+1} will be considered as a solution of the problem:

$$\langle f'(y^k), x \rangle \to \max, \qquad x \in D.$$

It is clear that:

$$\Pi(y^k) = \langle f'(y^k), x^{k+1} - y^k \rangle.$$

Step 3. If $\theta(x^k) \leq 0$ then set $x^* = x^k$ and stop.
Step 4. Otherwise, set $k = k + 1$, and go to step 2. □

Now we want to prove that algorithm 1 converges to a global maximum of problem (2).

THEOREM 4. *The sequence of points $\{x^k\}$ produced by the above mentioned algorithm is a maximizing sequence of problem (2), i.e.,*

$$\lim_{k \to \infty} f(x^k) = f^*$$

and all the limit points of the sequence $\{x^k\}$ are global maximizers of problem (2).

Proof. Note that from the construction of $\{x^k\}$ we have

$$x^k \in D \quad \text{and} \quad f(x^k) \leq f^* \qquad \text{for each} \quad k = 0, 1, 2 \ldots.$$

Without loss of generality let

$$\theta(x^k) > 0 \qquad \text{for all} \quad k = 0, 1, 2 \ldots. \tag{10}$$

In fact, otherwise, there exists k such that $\theta(x^k) \leq 0$.

Then, by Theorem 2, we can conclude that x^k is a solution of problem (2) and the proof is complete. Suppose, to the contrary, that $\{x^k\}$ is not a maximizing sequence of problem (2), i.e.,

$$\lim_{k \to \infty} \sup f(x^k) < f^* = f(x^*). \tag{11}$$

Where x^* is a global maximizer of problem (2).

First we show that the sequence $\{f(x^k)\}$ is strictly monotonically increasing. By the definition of $\theta(x^k)$ we have

$$\theta(x^k) = \Pi(y^k) = \langle f'(y^k), x^{k+1} - y^k\rangle > 0. \tag{12}$$

By the convexity of f this implies that

$$f(x^{k+1}) - f(x^k) = f(x^{k+1}) - f(y^k) \geq \langle f'(y^k), x^{k+1} - y^k\rangle > 0. \tag{13}$$

Hence, we obtain:

$$f(x^{k+1}) > f(x^k), \qquad \text{for all} \quad k = 0, 1, 2 \dots.$$

On the other hand the sequence $\{f(x^k)\}$ is bounded from above by the value of f^*, so that there exists a limit A:

$$\lim_{k\to\infty} f(x^k) = A.$$

Then, recalling (12) and (13), we obtain

$$\lim_{k\to\infty} \theta(x^k) = 0.$$

Now we introduce the following closed and convex sets:

$$C_k = \left\{x \in \mathrm{R}^n | f(x) \leq f(x^k)\right\}, \qquad \text{for all} \quad k = 0, 1, \dots.$$

It is clear that $x^* \notin C_k$, then there exists the projection of the point x^* on C_k, such that:

$$\exists u^k \in C_k : \|u^k - x^*\| = \inf_{x\in C_k} \|x - x^*\|$$

and

$$\|u^k - x^*\| > 0. \tag{14}$$

We also can consider u^k as the solution of the following convex programming problem:

$$g(x) = \frac{1}{2}\|x - x^*\|^2 \to \min, \quad x \in C_k.$$

Then the optimality condition for this problem at the point u^k looks like:

$$\begin{cases} \exists\lambda_0 \geq 0,\ \lambda_k \geq 0,\ \lambda_0^2 + \lambda_k^2 \neq 0 \\ \lambda_0 g'(u^k) + \lambda_k f'(u^k) = 0 \\ \lambda_k(f(u^k) - f(x^k)) = 0 \end{cases} \qquad \text{for all} \quad k = 0, 1, 2, \dots. \tag{15}$$

Now we shall show that $\lambda_0 \neq 0$ and $\lambda_k \neq 0$. It fact, if $\lambda_0 = 0$ then by (15), it follows that $\lambda_k > 0$ and $f'(u^k) = 0, \quad f(u^k) = f(x^k)$. This contradicts that $x^k \neq \arg\min_{x\in\mathrm{R}^n} f(x)$ for each $k = 0, 1, 2, \dots$.

Analogously, we show that $\lambda_k > 0$.

Since $g'(u^k) \neq 0$, the case $\lambda_k = 0$ is also impossible so that we can put $\lambda_0 = 1$ and $\lambda_k > 0$.

Then we can write (15) as follows:

$$u^k - x^* + \lambda_k f'(u^k) = 0. \tag{16}$$

Thus we have

$$\lambda_k = \frac{\|u^k - x^*\|}{\|f'(u^k)\|}. \tag{17}$$

On the other hand, by the definition of $\theta(x^k)$, it follows that:

$$\langle f'(u^k), x^* - u^k\rangle \leq \theta(x^k). \tag{18}$$

Using (16), (17) and (18) we have:

$$\|f'(u^k)\|\|u^k - x^*\| \leq \theta(x^k). \tag{19}$$

From the construction of $\{x^k\}$ the sequence $\{u^k\}$ is such that:

$$f(x^k) = f(u^k) \quad \text{and} \quad f(u^k) > f(u^{k-1}) \qquad \text{for all} \quad k = 0, 1, 2, \ldots.$$

Then, by Theorem 3, we obtain

$$\exists \delta > 0 : \|f'(u^k)\| \geq \delta \qquad \text{for each} \quad k = 0, 1, 2, \ldots.$$

From (19), it follows that:

$$0 \leq \delta\|u^k - x^*\| \leq \theta(x^k).$$

Taking into account that $\lim_{k\to\infty} \theta(x^k) = 0$ we have:

$$\lim_{k\to\infty} u^k = x^*.$$

By continuity of f on R^n we conclude that

$$\lim_{k\to\infty} f(x^k) = \lim_{k\to\infty} f(u^k) = f(x^*). \tag{20}$$

This contradicts (11). This contradiction implies that the assumption that $\{x^k\}$ is not a maximizing sequence of problem (2) must be false.

Since D is a compact set, there exists a convergent subsequence which we re-lable $\{x^k\}$, such that $\lim_{k\to\infty} x^k = \bar{x}$. By (20), we obtain:

$$\lim_{k\to\infty} f(x^k) = f(\bar{x}) = f^*$$

which completes the proof of the theorem. □

4. The Numerical Experiments

In this section we consider problem (2) with a quadratic objective function. The problem considered has the form:

$$f(x) = \frac{1}{2}\langle Ax, x\rangle \to \max, \quad x \in D \subset \mathrm{R}^n,$$

where A is a symmetric positive definite matrix and the set D is simple.

We note that the efficiency of the algorithm proposed above depends on the solution of problem (9). In order to solve problem (9) we can use classical methods, such as a penalty function method [1], a method of generalized gradients [12], or a method of nondifferentiable optimization [3]. Since $f'(x) = Ax$ in our case, then the auxiliary function $\Pi(y)$ takes the following form:

$$\Pi(y) = \max_{x\in D}\langle Ay, x\rangle - \langle Ay, y\rangle.$$

Thus we can write problem (9) as follows:

$$\Pi(y) \to \max, \quad f(y) = f(x^k),$$

which is reduced to the problem of maximizing the convex function $\varphi(y)$:

$$\varphi(y) = \max_{x\in D}\langle Ax, y\rangle \to \max, \quad \frac{1}{2}\langle Ay, y\rangle = \alpha_k,$$

where $\alpha_k = \frac{1}{2}\langle Ax^k, x^k\rangle$

We can write this last problem in the form

$$\varphi(y) \to \max, \quad \langle Ay, y\rangle \le 2\alpha_k,$$

because the function $\varphi(y)$ attains its maximum on the boundary of the set of equality constraint: $\langle Ay, y\rangle = 2\alpha_k$. This problem may then be solved by the method proposed in [13].

To check the efficiency of the algorithm proposed above, some test problems have been considered. The proposed algorithm has been implemented on an IBM PC/386 microcomputer in Pascal 6.0. For example, we considered the problems of maximizing a quadratic convex function over the spheres, parallelepipeds and polyhedrons. The problems (P1)–(P5) were taken from [15,16,2,8]. The list of the test problems considered is the following:

(P1) $$\begin{cases} f(x) = 2x_1^2 + 4x_2^2 - 5x_1x_2 \to \max \\ 0 \le x_1 \le 1, \quad 0 \le x_2 \le 1 \end{cases}$$

(P2)
$$\begin{cases} f(x) = x_1^2 + x_2^2 + x_3^2 + (x_3 - x_4)^2 \to \max \\ -2.3 \le x_i \le 2.7, \quad i = 1,2,3,4 \end{cases}$$

(P3)
$$\begin{cases} f(x) = \exp^{(2x_1 - x_2)^2} + x_1^2 + x_2^2 - 4x_1 - 4x_2 \to \max \\ 0 \le x_1 \le 1, \quad -2 \le x_2 \le 3. \end{cases}$$

(P4)
$$\begin{cases} x_1^2 + x_2^2 \to \max, \\ 4x_1 + 7x_2 \le 28 \\ x_1 - 5x_2 \le 5 \\ x_1 \ge 0, \quad x_2 \ge 0 \end{cases}$$

(P5)
$$\begin{cases} (x_1 - 1.2)^2 + (x_2 - 0.6)^2 \to \max \\ -2x_1 + x_2 \le 1 \\ x_1 + x_2 \le 4 \\ 0.5x_1 - x_2 \le 1 \\ 0 \le x_1 \le 3, \quad 0 \le x_2 \le 2 \end{cases}$$

(P6)
$$\begin{cases} x_1^2 + x_2^2 \to \max \\ -5x_1 + 13x_2 \le 72 \\ 11x_1 - 7x_2 \le 36 \\ 5x_1 - 9x_2 \le 28 \\ -11x_1 + 9x_2 \le 56 \end{cases}$$

(P7)
$$\begin{cases} 4(x_1 - 1)^2 + 25(x_2 - 2)^2 \to \max \\ 8.3x_1 + 20.5x_2 \le 170.15 \\ -7.5x_1 + 18x_2 \le 135 \\ -10.5x_1 + 7.7x_2 \le 80.85 \\ -3.7x_1 - 10.2x_2 \le 37.74 \\ -2.7x_1 - 13x_2 \le 35.1 \\ 4.5x_1 - 7x_2 \le 31.5 \\ -20 \le x_1 \le 20, \quad -20 \le x_2 \le 20. \end{cases}$$

Further, we considered problems

(P8)
$$\begin{cases} \|x\|^2 \to \max \\ -(n - i + 1) \le x_i \le n + 0.5i \qquad i = 1, 2, \ldots, n \end{cases}$$

(P9) $$\begin{cases} \|x\|^2 \to \max, \quad x \in B \\ B = \{x \in \mathrm{R}^n | \|x - c\| \le r\}, \quad c \neq 0, \quad r > 0 \end{cases}$$

(P10) $$\begin{cases} \sum_{i=1}^{n}(n - 1 - 0.1i)x_i^2 \to \max \\ -1 - i \le x_i \le 1 + 5i, \qquad i = 1, 2, \dots, n \end{cases}$$

(P11) $$\begin{cases} \|x - c\|^2 \to \max \\ -i \le x_i \le [\frac{n}{i}], \qquad i = 1, 2, \dots, n \end{cases}$$

(P12) $$f(x) = \langle Cx, x\rangle \to \max, \qquad x \in \Pi,$$

where $\Pi = \{x \in \mathrm{R}^n / -(n - i + 1) \le x_i \le n + 0.5i, \quad i = 1, 2, \dots, n\}$

$$C = \begin{bmatrix} n & n-1 & n-2 & \dots & 2 & 1 \\ n-1 & n & n-1 & \dots & 3 & 2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 2 & 3 & \dots & n-1 & n \end{bmatrix}.$$

The results of the numerical experiments for these problems are shown in Table I.

5. Conclusions

In this paper we have considered a class of global optimization problems. We have proposed an algorithm for the solution of the problem of maximizing a convex function on a so-called 'simple' set. This algorithm has been shown to be convergent. The proposed algorithm was tested on a variety of test problems. Our algorithm was implemented in Pascal 6 and run on an IBM PC 386 computer.

Acknowledgements

The author would like to thank O.V. Vasilyev and A.S. Strekalovskii of Irkutsk University for useful discussions on this paper. The author also wants to thank the Editor and the referees for their constructive comments and remarks, which greatly helped to improve an earlier version of the paper.

TABLE I.

Problems	dimension of the problems n	initial value of f	global maximum of f	Computing time (min:sec)
P1	2	2	4	0:3.71
P2	4	27	46.87	0:3.74
P3	2	8965.2	8886119.52	0:4.21
P4	2	16	42.0974	0:5.29
P5	2	2.45	3.4	0:5.45
P6	2	98	162	0:5.18
P7	2	549.08	871.947	0:5.33
P8	10	385	1646.25	0:15
P8	30	1612	43313.75	0: 2.57
P8	70	7000	546148.75	0: 3.56
P9	50	100	3025	0:4.79
P9	70	140	4235	0:5.1
P9	100	25	625	0:8.14
P10	3	5.4	721.4	0:1.8
P10	10	84.5	83712	0:1.12
P10	30	823.5	6440531	0:4
P10	60	8357	101506747	0:21
P10	80	5996	319560716	0:47
P10	100	9395	778330545	1:30
P10	150	21217.5	3927744505	4:56
P11	3	12.5	50	0:1
P11	5	103.75	165.25	0:2
P11	10	640	1297.75	0:1.1
P11	20	1280	11219	0:2
P11	50	3200	187732.75	0:9
P11	70	4480	522700.25	0:20
P11	90	5760	1120167.75	0:27
P11	100	6400	1541089	0:34
P11	200	12800	12494676.5	0:43
P11	400	25600	100639351.5	2.20
P11	500	32000	196830439	4:32
P12	2	12.8	45.5	0:1
P12	5	85	3604	0:81
P12	10	670	109333.5	0:3
P12	30	18010	25766625.5	1:12
P12	40	42680	108196334	2:33
P12	70	228690	1767930209	2:45
P12	80	341360	3444342668	3:50
P12	90	486030	6203290501.5	5:26
P12	99	646899	9986343609	7:26

I. Results of the numerical experiments for the problems (P1)–(P12).

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